Cutting-Planes for the Max-Cut Problem

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Outline

- **Literature review**
  - Max-Cut problem
  - ILP and IQP formulations
  - Product-type inequalities

- **Separation algorithms**
  - Triangle inequalities
  - Odd-clique and rounded-psd inequalities
  - Gap inequalities

- **Some further ‘tricks’**
  - Strengthening of gap inequalities
  - Stabilisation
  - A ‘special’ triangle packing

- **Computational results**

- **Conclusions**
The Max-cut problem is a well-known, fundamental and strongly $\mathcal{NP}$-hard combinatorial optimisation problem.

**Definition**

Let $G = (V, E)$ be an edge-weighted undirected graph. Max-cut calls for the vertex set to be partitioned into two subsets $(S, V \setminus S)$, in such a way that the sum of the weights on the edges crossing from $S$ to its complement $V \setminus S$ is maximised (i.e., looks for the cut $\delta(S)$ of maximal weight).
Example
Example (cont.)
Example (cont.)

Literature Review
Separation Algorithms
Some further tricks
Computational results
Conclusions
Some well known facts

- Surprisingly many **applications** (physics, VLSI, scheduling)
- Strongly $\mathcal{NP}$-hard (Karp *et al.* 1972, Johnson & Stockmeyer 1975)
- Polynomial solvable cases are known (e.g., planar graphs)
- 0.878 approximation-algorithm by Goemans & Williamson 1995
- W.l.o.g. one can always assume the **graph is complete**
A 0-1 Linear Programming formulation

\[
\begin{align*}
\text{max} & \quad \sum_{1 \leq i < j \leq n} w_{ij}x_{ij} \\
\text{s.t.} & \quad x_{ij} + x_{ik} + x_{jk} \leq 2 & (1 \leq i < j < k \leq n) \\
& \quad x_{ij} - x_{ik} - x_{jk} \leq 0 & (1 \leq i < j \leq n; k \neq i, j) \\
& \quad x_{ij} \in \{0, 1\} & (1 \leq i < j \leq n)
\end{align*}
\]

The convex hull in \( \mathbb{R}^{\binom{n}{2}} \) of solutions is called the cut polytope and often denoted by \( \text{CUT}_n \).
Remark

*It turns out that many of the known valid inequalities are of product type:*

\[ \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq r \]

*for some reals \( b_1, \ldots, b_n \) and some real \( r \).*
Triangle inequalities

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\text{s.t.} & \quad x_{ij} + x_{ik} + x_{jk} \leq 2 \quad (1 \leq i < j < k \leq n) \quad (1) \\
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- Inequalities (1)-(2) are called triangle inequalities
Triangle inequalities

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- Inequalities (1)-(2) are called triangle inequalities
- Triangle inequalities induce facets of \(\text{CUT}_n\) for all \(n\)
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- The polytope defined by triangle inequalities is called \textit{metric} polytope and denoted by MET$_n$
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\]

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- Triangle inequalities induce facets of CUT\(_n\) for all \(n\).
- The polytope defined by triangle inequalities is called metric polytope and denoted by MET\(_n\).
- Triangle inequalities provide a complete description for \(n \leq 4\).
- Triangle inequalities are of product-type.
An Integer Quadratic Programming formulation

Let \( y_i = \begin{cases} 
1 & \text{if } i \in S \\
-1 & \text{if } i \in V \setminus S 
\end{cases} \)

So, \( y_i \) takes the value 1 if vertex \( i \) is on one shore of the cut, and \(-1\) if it is on the other. Then the max-cut problem can be formulated as the following integer quadratic program:

\[
\begin{align*}
\max & \quad \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij}(1 - y_i y_j) \\
\text{s.t.} & \quad y_i \in \{-1, +1\} \quad (i \in V)
\end{align*}
\]
If we define an $n \times n$ matrix $Y$ in which, for all $i, j$, the entry $Y_{ij}$ represents the product $y_i y_j$, then the max-cut problem can be formulated as a semidefinite program:

$$\begin{align*}
\max & \quad \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} (1 - Y_{ij}) \\
\text{s.t.} & \quad Y_{ii} = 1 \quad (i \in V) \\
& \quad \text{rank}(Y) = 1 \\
& \quad Y \in S^n_+
\end{align*}$$

The SDP relaxation is derived by dropping the $\text{rank}(Y) = 1$ constraint.
The feasible region of the SDP relaxation called the *elliptope* is *convex, but not polyhedral*. 
Positive Semidefinite (psd) inequalities

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Laurent & Poljak 1995 showed that it can be \textit{projected} onto the space of the $x_{ij}$ variables via the mapping $Y_{ij} = 1 - 2x_{ij}$. 
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The projection of the ellipsoid is defined by the following positive semidefinite (psd) inequalities:

$$
\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \sigma(b)^2 / 4 \quad (\forall b \in \mathbb{R}^n),
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(3)

where $\sigma(b)$ denotes $\sum_{i=1}^n b_i$. 

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(In fact, we can assume that $\sigma(b)$ is odd without losing any important inequalities.) This leads to the following rounded-psd inequalities:

$$
\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \lfloor \sigma(b)^2 / 4 \rfloor \quad (\forall b \in \mathbb{Z}^n : \sigma(b) \text{ odd}).
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(4)
Rounded-psd and Odd-Clique inequalities

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$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \lceil \sigma(b)^2 / 4 \rceil \quad (\forall b \in \mathbb{Z}^n : \sigma(b) \text{ odd}).$$

(4)

In the special case where $b \in \{-1, 0, +1\}^n$, the rounded-psd inequalities reduce to the odd-clique inequalities (Barahona & Mahjoub).
Gap Inequalities

The rounded psd inequalities are formed by taking a psd inequality and reducing the right hand side.

Taking this idea further, we can reduce the right hand side until the constraint becomes supporting.

This leads to the gap inequalities of Laurent & Poljak 1996, which take the form:

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \left( \sigma(b)^2 - \gamma(b)^2 \right) / 4 \quad (\forall b \in \mathbb{Z}^n),$$

(5)

where $\gamma(b) := \min \{|z^T b| : z \in \{\pm 1\}^n\}$ is the so-called gap of $b$. 
Gap Inequalities: good news

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\[ \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \left( \sigma(b)^2 - \gamma(b)^2 \right) / 4 \quad (\forall b \in \mathbb{Z}^n) \]
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\[ \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \lfloor \sigma(b)^2 / 4 \rfloor \text{ and } \sigma(b) \text{ odd } (\forall b \in \mathbb{Z}^n) \]

gap → rounded psd
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gap $\rightarrow$ rounded psd $\rightarrow$ psd $\rightarrow$ gap-0
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gap $\longrightarrow$ rounded psd $\longrightarrow$ psd $\longrightarrow$ gap-0 $\longrightarrow$ negative-type
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\[ \sum_{1 \leq i < j \leq n} b_ib_jx_{ij} \leq \left( \sigma(b)^2 - \gamma(b)^2 \right) / 4 \quad (\forall b \in \mathbb{Z}^n) \]

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- They are infinite in number
- Is not known if they define a polyhedral set
- Computing the gap is $\mathcal{NP}$ - hard (Laurent & Poljak, 1997)
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Thus they have received little attention so far...
Triangle inequalities can be separated in $O(n^3)$ time by mere enumeration.

The following Algorithm 1 still runs in $O(n^3)$ time, but has the advantage of generating only $O(n^2)$ inequalities.

**Algorithm 1**: Heuristic separation algorithm for Triangle inequalities.

```plaintext
Input: A point $x^* \in \mathbb{R}^{n\choose 2}$ to be separated.
Output: FAILURE or a violated triangle inequality.

1. for $1 \leq i < j \leq n$ do
   2. if $x^*_{ij} > (2 + \epsilon)/3$ then
      3. Find $k : x^*_{ik} + x^*_{jk} = \max_{h=j+1...n} \{x^*_i + x^*_j\}$;
      4. if $x^*_{ij} + x^*_{ik} + x^*_{jk} \geq 2 + \epsilon$ then
         5. Return violated inequality $x^*_{ij} + x^*_{ik} + x^*_{jk} \leq 2$;
      6. end
   7. end
   8. if $x^*_{ij} > \epsilon$ then
      9. Find $k : x^*_{ik} + x^*_{jk} = \min_{h=1...n} \{x^*_i + x^*_j\}$;
     10. if $x^*_{ij} - x^*_{ik} - x^*_{jk} \geq \epsilon$ then
         11. Return violated inequality $x^*_{ij} - x^*_{ik} - x^*_{jk} \leq 0$;
     12. end
   13. end
14. end
```
Separation of Odd-Clique Inequalities

Using the mapping, the odd clique inequalities can be written as:

\[ b^T Y b \geq 1 \quad (\forall b \in \{0, \pm 1\}^n : \sigma(b) \text{ odd}). \]

If \( Y^* \) (the point to be separated) is \text{psd}, then the separation problem reduces to the following Convex Integer Quadratic Programme (CIQP):

\[
\min \left\{ b^T Y^* b : \sigma(b) = 2k + 1, \ b \in \{0, \pm 1\}^n, \ k \in \mathbb{Z}_+ \right\}.
\]

There is good software available for solving CIQPs.

If \( Y^* \) is not \text{psd}, then one must use a slightly different approach...
The odd-clique heuristic of Helmberg & Rendl 1998 is a simple greedy:

- The basic idea consists in keeping collection of odd clique (including triangle) inequalities from which new odd clique inequalities can be derived.
- For each inequality in the collection, the algorithm tries to extend the clique by inserting two nodes in the hope of obtaining a new violated odd clique inequality, that is added to the current pool.
- The advantage of this heuristic is that the inequalities start out very sparse and get denser as the algorithm progresses.
Similarly to the case of odd-clique inequalities, rounded-psd inequalities can be separated exactly via CIQP.

Another approach is to use a greedy heuristic, similar to the one of Helmberg & Rendl 1998 for odd-clique inequalities.

- Take a previously-generated round psd inequality (which could be a triangle or an odd-clique inequality)
- Test whether a violated rounded psd inequality can be obtained by incrementing or decrementing two of the components of the $b$ vector.
We present now an $O(n^2 u)$ (where $u$ is an UB on $\|b\|_1 = \sum_{i=1}^{n} |b_i|$) heuristic separation scheme for gap inequalities.

It is based on the spectral decomposition of the current solution $Y^*$. 
Let $u$ be a prespecified upper bound on $\|b\|_1$
Heuristic separation of Gap Inequalities (cont.)

- Let $u$ be a prespecified upper bound on $\|b\|_1$
- Construct the matrix $Y^*$
Heuristic separation of Gap Inequalities (cont.)

- Let \( u \) be a prespecified upper bound on \( \|b\|_1 \)
- Construct the matrix \( Y^* \)
- Let \( b^* \in \mathbb{R}^n \) be an eigenvector of \( Y^* \) corresponding to the minimum eigenvalue
Heuristic separation of Gap Inequalities (cont.)

- Let $u$ be a prespecified upper bound on $\|b\|_1$
- Construct the matrix $Y^*$
- Let $b^* \in \mathbb{R}^n$ be an eigenvector of $Y^*$ corresponding to the minimum eigenvalue
- Compute $u^* = \|b^*\|_1$
Let $u$ be a prespecified upper bound on $||b||_1$

Construct the matrix $Y^*$

Let $b^* \in \mathbb{R}^n$ be an eigenvector of $Y^*$ corresponding to the minimum eigenvalue

Compute $u^* = ||b^*||_1$

Scale $b^*$ by multiplying it by $u/u^*$
Heuristic separation of Gap Inequalities (cont.)

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- Scale $b^*$ by multiplying it by $u/u^*$
- Round the components of $b^*$ to integers
- Compute the gap of $b^*$ and check the corresponding gap inequality for violation.
Note that

\[ \gamma(b) = \|b\|_1 - 2 \text{ SSP}, \]

where SSP is the solution to the following subset-sum problem:

\[
\max \left\{ \sum_{i=1}^{n} |b_i|y_i : \sum_{i=1}^{n} |b_i|y_i \leq \left\lfloor \frac{\|b\|_1}{2} \right\rfloor, \ y \in \{0, 1\}^n \right\}.
\]

This subset-sum problem can be solved in \( O(n\|b\|_1) \) time by dynamic programming.
Recall that psd inequalities constitute a relaxation of gap inequalities (i.e., $\gamma(b)^2$ is removed).
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Obtaining the gap $\gamma(b)$, calls for solving an instance of the *subset sum problem* (SSP) and setting $\gamma(b) = \|b\|_1 - 2SSP$.
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Obtaining the gap $\gamma(b)$, calls for solving an instance of the *subset sum problem* (SSP) and setting $\gamma(b) = ||b||_1 - 2SSP$ ...can we do better than that?

We can *adjust the vector $b^*$* and try to maximise the violation of the resulting gap inequality (strengthening).
For $r = 0, \ldots, u$, let

$$f(r) = \max \left\{ \sum_{j \neq i} |b_j|y_j : \sum_{j \neq i} |b_j|y_j \leq r, \; y \in \{0, 1\}^n \right\}$$

Set $b' = \begin{cases} \frac{b_j}{b_i + \Delta} & \text{if } j \neq i \\ b_i + \Delta & \text{if } j = i \end{cases}$

note that $\|b'\|_1 = \|b\|_1 + |b_i + \Delta_i| - b_i$

$\text{GAP}' : \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} + \Delta \sum_{j \neq i} b_j x_{ij} \leq \left( \sigma(b')^2 - \gamma(b')^2 \right) / 4 \quad (\forall b \in \mathbb{Z}^n)$
Strengthening of Gap Inequalities (cont.)

For \( r = 0, \ldots, u \), let

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f(r) = \max \left\{ \sum_{j \neq i} |b_j| y_j : \sum_{j \neq i} |b_j| y_j \leq r, \ y \in \{0, 1\}^n \right\}
\]

Set \( b' = \begin{cases} b_j & \text{if } j \neq i \\ b_i + \Delta & \text{if } j = i \end{cases} \) note that \( ||b'||_1 = ||b||_1 + |b_i + \Delta| - b_i \)

\[
\text{GAP'}: \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} + \Delta \sum_{j \neq i} b_j x_{ij} \leq \left( \sigma(b')^2 - \gamma(b')^2 \right) / 4 \quad (\forall b \in \mathbb{Z}^n)
\]

\[
\text{SSP'} = \max |b_i + \Delta| y_i + \sum_{k \neq i} |b_k| y_k \\
\text{s.t.} \quad |b_i + \Delta| y_i + \sum_{k \neq i} |b_k| y_k \leq \left\lfloor \frac{||b'||_1}{2} \right\rfloor \\
y \in \{0, 1\}^n
\]

That is \( \text{SSP'} = \max \left\{ f \left( \left\lfloor \frac{||b'||_1}{2} \right\rfloor \right), |b_i + \Delta| + f \left( \left\lfloor \frac{||b'||_1}{2} \right\rfloor - |b_i + \Delta| \right) \right\} \)

Set \( \gamma(b') = ||b'||_1 - 2\text{SSP'} \) and export the stronger gap inequality
The tailing-off phenomenon leads to very long running times for large instances. One way around it is using primal stabilisation as follows:

- Solve the SDP relaxation, yielding a solution $x^*$
Primal Stabilisation

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1. Solve the SDP relaxation, yielding a solution $x^*$
2. Add to the LP a lower and upper bound for each variable, to force the LP solution to stay near $x^*$
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1. Solve the SDP relaxation, yielding a solution $x^*$
2. Add to the LP a lower and upper bound for each variable, to force the LP solution to stay near $x^*$
3. Run a cutting-plane algorithm
4. If no more psd inequalities are violated (within some tolerance), or if none of the lower and upper bounds are binding, remove the lower and upper bounds
To improve the initial LP relaxation we generate a collection of special triangle inequalities.

Special means:

- likely to substantially improve the initial upper bound
- not likely to slow down the LP solver
To improve the initial LP relaxation we generate a collection of special triangle inequalities. Special means:

- likely to substantially improve the initial upper bound
- not likely to slow down the LP solver

Property 1 suggests that the more TIs we use, the better. But property 2 suggests that it would be a good idea to ensure that no variable appears in more than one of the chosen TIs.
To improve the initial LP relaxation we generate a collection of special triangle inequalities. Special means:

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For each triangle in the packing, we have four TIs to choose from. So we select the best one according to the signs of the weights of the corresponding edges.
We summarise the results obtained from the application of two cutting plane schemes on a subset of Max-Cut instances taken from the Biq Mac Library\footnote{http://BiqMac.uni-klu.ac.at}. 
Computational results

We summarise the results obtained from the application of two cutting plane schemes on a subset of Max-Cut instances taken from the Biq Mac Library\(^3\).

- Among these instances, \texttt{g.05.n} are unweighted with edge probability 0.5 and \(n = 60, 80, 100\). Instances \texttt{pm1d.80.0}, have edge probability 0.99, \(w_{ij} = \{-1, 0, 1\}\) for \(1 \leq i \leq j \leq n\) and \(n = 80\).

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- Each row is the average of 10 instances of the given class.

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### Computational results (cont.)

<table>
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<th></th>
<th>% IG TIs</th>
<th>% IG SDP</th>
<th>% IG</th>
<th>Time (scs)</th>
<th>% IG</th>
<th>Time (scs)</th>
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<td>22.3</td>
<td>35.55</td>
<td>180</td>
<td>15.12</td>
<td>3650</td>
</tr>
</tbody>
</table>

**Table:** Integrality gaps of various relaxations of the max-cut problem.
Conclusions

- Our preliminary computations (table 1) show that our cutting-plane scheme based on the heuristic separation of GAP and triangle inequalities is capable of beating the SDP relaxation on all instances.
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- Odd clique and rounded psd inequalities generally do not help in terms of bound quality, but can be useful to reduce the number of triangle inequalities in the LP.

- Finally, the results presented refer to GAP inequalities **without strengthening**. In fact, after many experiments, we noticed that the best set-up for our cutting-plane scheme, is separating ‘rounds' of GAP inequalities, by generating one inequality for each negative eigenvalue in the $Y$ matrix.
Ta!