Cutting-Planes for the Max-Cut Problem

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Outline

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  - Max-Cut problem
  - ILP and IQP formulations
  - Product-type inequalities
- Separation algorithms
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  - Odd-clique and rounded-psd inequalities
  - Gap inequalities
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  - Strengthening of gap inequalities
  - Stabilisation
  - A ‘special’ triangle packing
- Computational results
- Conclusions

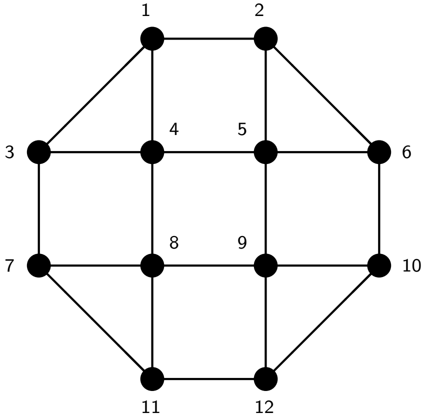
The Max-Cut problem

The Max-cut problem is a well-known, fundamental and strongly $\mathcal{NP}$-hard combinatorial optimisation problem.

Definition

Let $G = (V, E)$ be an edge-weighted undirected graph. Max–cut calls for the vertex set to be partitioned into two subsets $(S, V \setminus S)$, in such a way that the sum of the weights on the edges crossing from S to its complement $V \setminus S$ is maximised (i.e., looks for the cut $\delta(S)$ of maximal weight).

Example



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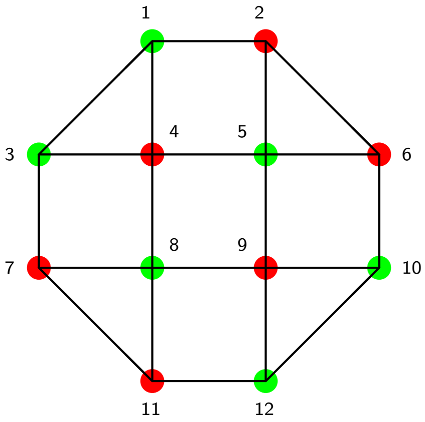

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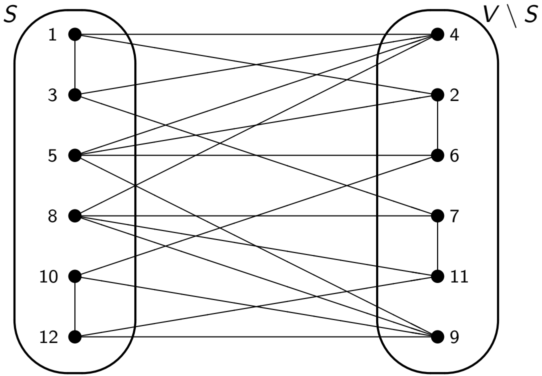

Some well known facts

- Surprisingly many applications (physics, VLSI, scheduling)
- Strongly $\mathcal{NP}$-hard (Karp *et al.* 1972, Johnson & Stockmeyer 1975)
- Polynomial solvable cases are known (e.g., planar graphs)
- 0.878 approximation-algorithm by Goemans & Williamson 1995
- W.l.o.g. one can always assume the graph is complete

A 0-1 Linear Programming formulation

$$
\begin{array}{rll}
\max & \sum_{1 \le i < j \le n} w_{ij} x_{ij} & \\
\text{s.t.} & x_{ij} + x_{ik} + x_{jk} \le 2 & (1 \le i < j < k \le n) \\
& x_{ij} - x_{ik} - x_{jk} \le 0 & (1 \le i < j \le n;\, k \ne i, j) \\
& x_{ij} \in \{0, 1\} & (1 \le i < j \le n)
\end{array}
$$

The convex hull in $\mathbb{R}^{\binom{n}{2}}$ of solutions is called the cut polytope and often denoted by CUT_n.

Product-type Inequalities

Remark

It turns out that many of the known valid inequalities are of product type:

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq r$$

for some reals $b_1, \ldots, b_n$ and some real r.

Triangle inequalities

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\begin{array}{rll}
\max & \sum_{1\le i<j\le n} w_{ij}x_{ij} & \\
\text{s.t.} & x_{ij}+x_{ik}+x_{jk}\le 2 \quad (1\le i<j<k\le n) & (1)\\
& x_{ij}-x_{ik}-x_{jk}\le 0 \quad (1\le i<j\le n;\, k\ne i,j) & (2)\\
& x_{ij}\in\{0,1\} \quad (1\le i<j\le n) &
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- Inequalities (1)-(2) are called triangle inequalities

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- Inequalities (1)-(2) are called triangle inequalities
- Triangle inequalities induce facets of CUT_n for all n
- The polytope defined by triangle inequalities is called metric polytope and denoted by MET_n
- Triangle inequalities provide a complete description for $n \le 4$
- Triangle inequalities are of product-type

An Integer Quadratic Programming formulation

Let $y_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \in V \setminus S \end{cases}$

So, y_i takes the value 1 if vertex i is on one shore of the cut, and -1 if it is on the other. Then the max-cut problem can be formulated as the following integer quadratic program:

$$\begin{array}{lll} \max & \frac{1}{2} \sum_{1 \le i < j \le n} w_{ij}(1 - y_i y_j) & \\ \text{s.t.} & y_i \in \{-1, +1\} & (i \in V) \end{array}$$

SDP relaxation

If we define an $n \times n$ matrix Y in which, for all i, j, the entry Y_{ij} represents the product $y_i y_j$, then the max-cut problem can be formulated as a semidefinite program:

$$
\begin{array}{lcr}
\max & \frac{1}{2}\sum_{1 \le i < j \le n} w_{ij}(1 - Y_{ij}) & \\
\text{s.t.} & Y_{ii} = 1 & (i \in V) \\
 & rank(Y) = 1 & \\
 & Y \in \mathcal{S}_+^n &
\end{array}
$$

The SDP relaxation is derived by dropping the $rank(Y) = 1$ constraint.

Positive Semidefinite (psd) inequalities

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The projection of the elliptope is defined by the following positive semidefinite (psd) inequalities:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \sigma(b)^2/4 \qquad (\forall b \in \mathbb{R}^n), \tag{3}$$

where $\sigma(b)$ denotes $\sum_{i=1}^n b_i$.

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Psd-inequalities are of product-type

Rounded-psd and Odd-Clique inequalities

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(In fact, we can assume that $\sigma(b)$ is odd without losing any important inequalities.) This leads to the following rounded-psd inequalities:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \lfloor \sigma(b)^2/4 \rfloor \qquad (\forall b \in \mathbb{Z}^n : \sigma(b) \text{ odd}). \tag{4}$$

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In the special case where $b \in \{-1, 0, +1\}^n$, the rounded-psd inequalities reduce to the odd-clique inequalities (Barahona & Mahjoub) .

Gap Inequalities

The rounded psd inequalities are formed by taking a psd inequality and reducing the right hand side.

Taking this idea further, we can reduce the right hand side until the constraint becomes supporting.

This leads to the gap inequalities of Laurent & Poljak 1996, which take the form:

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \left(\sigma(b)^2 - \gamma(b)^2 \right) / 4 \qquad (\forall b \in \mathbb{Z}^n), \tag{5}$$

where $\gamma(b) := \min \left\{ |z^T b| : z \in \{\pm 1\}^n \right\}$ is the so-called gap of b.

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\text{gap} \arrow[r] & \text{rounded psd} \arrow[r] \arrow[d] & \text{psd} \arrow[r] & \text{gap-0} \arrow[r] & \text{negative-type} \\
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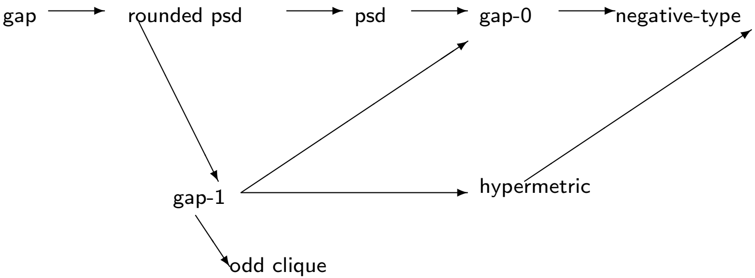

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Thus they have received little attention so far...

Separation of Triangle Inequalities

Triangle inequalities can be separated in $\mathcal{O}(n^3)$ time by mere enumeration.

The following Algorithm 1 still runs in $\mathcal{O}(n^3)$ time, but has the advantage of generating only $\mathcal{O}(n^2)$ inequalities.

Algorithm 1: Heuristic separation algorithm for Triangle inequalities.

Input: A point $x^* \in \mathbb{R}^{\binom{n}{2}}$ to be separated.
Output: Failure or a violated triangle inequality.

for $1 \leq i < j \leq n$ **do**
  if $x^*_{ij} > (2+\epsilon)/3$ **then**
    Find $k : x^*_{ik} + x^*_{jk} = \max_{h=j+1\dots n}\{x^*_{ih} + x^*_{jh}\}$;
    if $x^*_{ij} + x^*_{ik} + x^*_{jk} \geq 2 + \epsilon$ **then**
      Return violated inequality $x_{ij} + x_{ik} + x_{jk} \leq 2$;
    end
  end
  if $x^*_{ij} > \epsilon$ **then**
    Find $k : x^*_{ik} + x^*_{jk} = \min_{h=1\dots n}\{x^*_{ih} + x^*_{jh}\}$;
    if $x^*_{ij} - x^*_{ik} - x^*_{jk} \geq \epsilon$ **then**
      Return violated inequality $x_{ij} - x_{ik} - x_{jk} \leq 0$;
    end
  end
end

Separation of Odd-Clique Inequalities

Using the mapping, the odd clique inequalities can be written as:

$$b^T Y b \geq 1 \qquad (\forall b \in \{0, \pm 1\}^n : \sigma(b) \text{ odd}).$$

If Y^* (the point to be separated) is psd, then the separation problem reduces to the following Convex Integer Quadratic Programme (CIQP):

$$\min \left\{ b^T Y^* b : \ \sigma(b) = 2k + 1, \ b \in \{0, \pm 1\}^n, \ k \in \mathbb{Z}_+ \right\}.$$

There is good software available for solving CIQPs.

If Y^* is not psd, then one must use a slightly different approach...

Separation of Odd-Clique Inequalities (cont.)

The odd-clique heuristic of Helmberg & Rendl 1998 is a simple greedy:

- The basic idea consists in keeping collection of odd clique (including triangle) inequalities from which new odd clique inequalities can be derived
- For each inequality in the collection, the algorithm tries to extend the clique by inserting two nodes in the hope of obtaining a new violated odd clique inequality, that is added to the current pool.
- The advantage of this heuristic is that the inequalities start out very sparse and get denser as the algorithm progresses.

Separation of Rounded-psd Inequalities

Similarly to the case of odd-clique inequalities, rounded-psd inequalities can be separated exactly via CIQP.

Another approach is to use a greedy heuristic, similar to the one of Helmberg & Rendl 1998 for odd-clique inequalities.

- Take a previously-generated round psd inequality (which could be a triangle or an odd-clique inequality)
- Test whether a violated rounded psd inequality can be obtained by incrementing or decrementing two of the components of the b vector.

Heuristic separation of Gap Inequalities

We present now an $\mathcal{O}(n^2 u)$ (where u is an UB on $||b||_1 = \sum_{i=1}^{n} |b_i|$) heuristic separation scheme for gap inequalities.

It is based on the spectral decomposition of the current solution Y^*.

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- Scale b^* by multiplying it by u/u^*
- Round the components of b^* to integers
- Compute the gap of b^* and check the correspondng gap inequality for violation.

Heuristic separation of Gap Inequalities (cont.)

Note that

$$\gamma(b) = ||b||_1 - 2\ \mathrm{SSP},$$

where SSP is the solution to the following subset-sum problem:

$$\max\left\{ \sum_{i=1}^{n} |b_i| y_i :\ \sum_{i=1}^{n} |b_i| y_i \leq \left\lfloor \frac{||b||_1}{2} \right\rfloor,\ y \in \{0,1\}^n \right\}.$$

This subset-sum problem can be solved in $\mathcal{O}(n||b||_1)$ time by dynamic programming.

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3. We can adjust the vector b^* and try to maximise the violation of the resulting gap inequality (strengthening).

Strengthening of Gap Inequalities (cont.)

1. For $r = 0, \ldots, u$, let

$$f(r) = \max \left\{ \sum_{j \neq i} |b_j| y_j : \sum_{j \neq i} |b_j| y_j \leq r, \ y \in \{0,1\}^n \right\}$$

2. Set $b' = \begin{cases} b_j & \text{if } j \neq i \\ b_i + \Delta & \text{if } j = i \end{cases}$ note that $||b'||_1 = ||b||_1 + |b_i + \Delta_i| - b_i$

   $\text{GAP}' : \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} + \Delta \sum_{j \neq i} b_j x_{ij} \leq \left(\sigma(b')^2 - \gamma(b')^2 \right) / 4 \qquad (\forall b \in \mathbb{Z}^n)$

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1. For $r = 0, \ldots, u$, let

$$f(r) = \max \left\{ \sum_{j \neq i} |b_j| y_j : \sum_{j \neq i} |b_j| y_j \leq r, \; y \in \{0,1\}^n \right\}$$

2. Set $b' = \begin{cases} b_j & \text{if } j \neq i \\ b_i + \Delta & \text{if } j = i \end{cases}$ note that $||b'||_1 = ||b||_1 + |b_i + \Delta_i| - b_i$

   $\text{GAP}' : \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} + \Delta \sum_{j \neq i} b_j x_{ij} \leq \left(\sigma(b')^2 - \gamma(b')^2 \right) /4 \qquad (\forall b \in \mathbb{Z}^n)$

$$\begin{aligned} SSP' = \max \quad & |b_i + \Delta| y_i + \textstyle\sum_{k \neq i} |b_k| y_k \\ \text{s.t.} \quad & |b_i + \Delta| y_i + \textstyle\sum_{k \neq i} |b_k| y_k \leq \left\lfloor \frac{||b'||_1}{2} \right\rfloor \\ & y \in \{0,1\}^n \end{aligned}$$

   That is $SSP' = \max \left\{ f\left(\left\lfloor \frac{||b'||_1}{2} \right\rfloor \right), \; |b_i + \Delta| + f\left(\left\lfloor \frac{||b'||_1}{2} \right\rfloor - |b_i + \Delta| \right) \right\}$

3. Set $\gamma(b') = ||b'||_1 - 2SSP'$ and export the stronger gap inequality

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4. If no more psd inequalities are violated (within some tolerance), or if none of the lower and upper bounds are binding, remove the lower and upper bounds

Edge-disjoint Triangle Packing

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For each triangle in the packing, we have four TIs to choose from. So we select the best one according to the signs of the weights of the corresponding edges.

Computational results

We summarise the results obtained from the application of two cutting plane schemes on a subset of Max-Cut instances taken from the Biq Mac Library [3].

[3] http://BiqMac.uni-klu.ac.at

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Computational results (cont.)

	% IG TIs	% IG SDP	Mult.GAPs		Mult.GAPs+TIs	
			% IG	Time (scs)	% IG	Time (scs)
g_05_60	10.83	2.52	5.31	60	1.84	275
g_05_80	13.26	2.18	4.81	180	1.59	2685
g_05_100	15.28	2.23	5.01	600	1.93	15383
pmd1_80	102.1	22.3	35.55	180	15.12	3650

Table: Integrality gaps of various relaxations of the max-cut problem.

Conclusions

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- Our preliminary computations (table 1) show that our cutting-plane scheme based on the heuristic separation of GAP and triangle inequalities is capable of beating the SDP relaxation on all instances
- Odd clique and rounded psd inequalities generally do not help in terms of bound quality, but can be useful to reduce the number of triangle inequalities in the LP
- Finally, the results presented refer to GAP inequalities without strengthening. In fact, after many experiments, we noticed that the best set-up for our cutting-plane scheme, is separating 'rounds' of GAP inequalities, by generating one inequality for each negative eigenvalue in the Y matrix.

Ta!