# Reformulating Mixed-Integer Quadratically Constrained Quadratic Programs

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**Abstract**

It is well known that semidefinite programming (SDP) can be used to derive useful relaxations for a variety of optimisation problems. Moreover, in the particular case of mixed-integer quadratic programs, SDP has been used to *reformulate* problems, rather than merely relax them. The purpose of reformulation is to strengthen the continuous relaxation of the problem, while leaving the optimal solution unchanged. In this paper, we explore the possibility of extending the reformulation approach to the (much) more general case of mixed-integer quadratically constrained quadratic programs.

**Keywords:** mixed-integer nonlinear programming, semidefinite programming, quadratically constrained quadratic programming

## 1 Introduction

It has been known for some time that semidefinite programming (SDP) can be used to derive strong tractable convex relaxations of hard optimisation problems. Examples of problems to which this idea has been successfully applied include (in order of increasing generality) *zero-one linear programming* (0-1 LP) [15], *zero-one quadratic programming* (0-1 QP) [7, 9, 17], non-convex *quadratically constrained quadratic programming* (QCQP) [6, 20, 23] and general *polynomial programming* [13, 19].

In a recent paper, Billionnet *et al.* [3] applied SDP to *equality-constrained 0-1 QP*, but in an unconventional way. They proposed to use SDP to *reformulate* such 0-1 QP instances, rather than merely *relax* them. Their method, which they called *Quadratic Convex Reformulation* (QCR), has two effects.

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First, it converts non-convex instances into convex ones. Second, when applied to instances that are already convex, it improves the bound obtained by solving the continuous relaxation of the instance.

The motivation behind the QCR method is that there now exist quite effective software packages for convex *mixed-integer quadratic programming* (MIQP). Once QCR has been applied to the 0-1 QP instance, the reformulated instance can simply be passed to such a software package, which is then treated as a 'black box'. In particular, if the software package solves the problem by branch-and-bound with convex QP relaxations, the improved bounds obtained by reformulation can be expected to lead to a reduction in the number of branch-and-bound nodes.

In a recent follow-up paper, Billionnet *et al.* [2] show that, under certain conditions, the QCR method can be extended from equality-constrained 0-1 QP to general MIQP. The purpose of the present paper is to show that, again under certain conditions, it can be extended to the even more general case of *mixed-integer quadratically constrained quadratic programming* (MIQCQP). Interestingly, handling quadratic constraints adequately turns out to be a non-trivial exercise.

We remark that software is now emerging for convex MIQCQP (see, e.g., Drewes [4]). Therefore, our extension of QCR is likely to be of practical use.

The structure of the paper is as follows. In Section 2, we review the relevant literature. In Section 3, we show that the QCR method can be adapted easily to deal with 0-1 QCQP instances in which all quadratic constraints are equations. In Section 4, we consider the case in which quadratic inequalities are also present. In Sections 5 and 6, we consider the further complications posed by integer and continuous variables, respectively. Finally, concluding remarks are made in Section 7.

# 2 Literature Review

We now review the relevant literature. In Subsection 2.1, we review certain semidefinite and Lagrangian relaxations for non-convex QCQP, which will be useful later. In Subsection 2.2, we recall some of the stronger relaxations that have been proposed for the special case of 0-1 QP. In Subsection 2.3, we briefly review the two papers on the QCR method [2, 3].

## 2.1 Relaxations of non-convex QCQP

A general instance of QCQP can be written in the following form:

$$\begin{array}{ll} \inf & x^T Q^0 x + c^0 \cdot x \\ \text{s.t.} & x^T Q^j x + c^j \cdot x \le h_j \quad (j = 1, \dots, m) \\ & x \in \mathbb{R}^n, \end{array} \qquad (1)$$

where the $Q^j$ are symmetric matrices of order $n$, the $c^j$ are $n$-vectors and the $h_j$ are scalars. (We write 'inf' rather than 'min' because it is possible that the infimum is not attainable.)

Now suppose that at least one of the $Q^j$ is not positive semidefinite (psd), so that the problem is not convex. We can derive a semidefinite relaxation as follows [6, 20, 23]. We define the $n \times n$ matrix $X = xx^T$, along with the augmented matrix

$$Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Note that $Y$ is symmetric and psd. The following SDP is therefore a relaxation of non-convex QCQP:

$$\begin{array}{ll} \inf & Q^0 \bullet X + c^0 \cdot x \\ \text{s.t.} & Q^j \bullet X + c^j \cdot x \le h_j \quad (j = 1, \ldots, m) \\ & Y \succeq 0. \end{array}$$

Here, $Q^j \bullet X$ denotes $\sum_{i=1}^n \sum_{k=1}^n Q^j_{ik} X_{ik}$, and $Y \succeq 0$ means that $Y$ is symmetric and psd.

There is a connection between semidefinite and Lagrangian relaxations of non-convex QCQP [5, 6, 14, 17]. Suppose we relax the constraints (1) in Lagrangian fashion, using a vector $\lambda \in \mathbb{R}^m_+$ of Lagrangian multipliers. The Lagrangian is

$$f(x, \lambda) \;=\; x^T \left( Q^0 + \sum_{j=1}^m \lambda_j Q^j \right) x + \left( c^0 + \sum_{j=1}^m \lambda_j c^j \right) \cdot x - \sum_{j=1}^m \lambda_j h_j.$$

The relaxed problem is:

$$\inf \left\{ f(x, \lambda) : \; x \in \mathbb{R}^n \right\},$$

which is an unconstrained quadratic minimisation problem. The Lagrangian dual is:

$$\sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} f(x, \lambda).$$

As explained in [6, 14, 17], if the supremum is attainable by some multiplier vector $\lambda^*$, then $\lambda^*$ must be an optimal dual solution to the SDP. If in addition the SDP or its dual satisfy the Slater condition, then the semidefinite and Lagrangian bounds will be equal. An analogous result holds when quadratic equations, rather than inequalities, are present.

## 2.2 Relaxations of 0-1 QP

An instance of 0-1 QP can be written in the following form:

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & Ax = b \qquad (2) \\ & Dx \le f \qquad (3) \\ & x \in \{0,1\}^n, \end{aligned}$$

where $Q$ is again a symmetric square matrix, $A$ and $D$ are matrices of appropriate dimension, and $c$, $b$ and $f$ are vectors of appropriate dimension.

As observed by many authors (e.g., [8, 12, 14, 15, 17]), the condition that $x$ be binary is equivalent to the non-convex quadratic constraints

$$x_i^2 - x_i = 0 \qquad (i = 1, \dots, n). \tag{4}$$

That is, 0-1 QP can be regarded as a special case of non-convex QCQP. This observation suggests immediately the following semidefinite relaxation:

$$\inf \left\{ Q \bullet X + c^T x : \ (2),\ (3),\ x = \mathrm{diag}(X),\ Y \succeq 0 \right\}.$$

The SDP can be strengthened using some ideas presented in [15, 22]. Given a linear equation in the system (2), say $a^j \cdot x = b_j$, and any variable, say $x_k$, the quadratic equation $(a^j \cdot x)x_k = b_j x_k$ is satisfied by all feasible solutions. This implies that the equation

$$\sum_{i=1}^{n} a_i^j X_{ik} - b_j x_k = 0 \tag{5}$$

can be added to the SDP. In a similar way, any linear inequality in the system (3) can be multiplied by either $x_k$ or $1 - x_k$ to yield valid quadratic inequalities, which can also be converted into valid inequalities for the SDP. Moreover, pairs of linear inequalities can be multiplied together to yield even more valid quadratic inequalities.

In [14, 17], it is suggested to add to the SDP the single additional equation

$$A^T A \bullet X = b^T b \tag{6}$$

instead of the equations (5). Remarkably, whether we add all of the equations (5), or just the single equation (6), the feasible region of the strengthened SDP is the same (Faye & Roupin [5]). A similar idea, but for the case of inequalities, appears in Roupin [21].

The SDP can be strengthened even further, by adding generic valid inequalities for quadratic 0-1 problems, such as triangle or hypermetric inequalities [9, 16]. Details are omitted for the sake of brevity.

It follows from the result mentioned in the previous subsection that, for any semidefinite relaxation of 0-1 QP, there is a corresponding Lagrangian relaxation. Examples of such relaxations appear, for example, in [6, 11, 12, 14, 17]. Again, details are omitted for brevity.

### 2.3 The QCR method

We now move from *relaxation* to *reformulation*, starting with the special case of 0-1 QP. For our purposes, *reformulating* a 0-1 QP instance means perturbing the objective function in such a way that the cost of each feasible solution is unchanged.

An early paper on this topic was Hammer & Rubin [8]. They proposed to convert non-convex 0-1 QP instances into convex ones simply by adding $\sum_{1 \le i \le n} M(x_i^2 - x_i)$ to the objective, where $M > 0$ is large enough to ensure that the matrix $Q + MI$ of quadratic coefficients is psd. (A suitable value for $M$ is the minimum eigenvalue of $Q$, multiplied by minus one.)

Billionnet *et al.* [3] develop this idea further, though only for equality-constrained 0-1 QP. They perturb the objective by adding terms of the form $\lambda_i(x_i^2 - x_i)$, where $\lambda \in \mathbb{R}^n$, along with terms of the form $\mu_{jk}(a^j \cdot x - b_j)x_k$, where $\mu$ is a real matrix of the same dimension as $A$. They then propose to select values for $\lambda$ and $\mu$ that (i) render the resulting 0-1 QP instance convex, and (ii) maximise the lower bound obtained by solving the continuous relaxation of the instance.

To do this, Billionnet *et al.* [3] propose to solve the SDP mentioned in the previous subsection, strengthened with the constraints of the form (5), and then set $\lambda$ and $\mu$ to the optimal dual values for the constraints $\text{diag}(X) = x$ and (5), respectively. (We remark that this assumes that such optimal dual values exist — see Subsection 3.2.) This is the original QCR method.

In the subsequent paper [2], Billionnet *et al.* extend the QCR method to the case of MIQP. The basic ideas are as follows:

- Each general-integer variable is replaced with a small number of binary variables, using the standard bit representation.
- The objective function is perturbed using the equations, but not the inequalities.
- Continuous variables can be handled simply by omitting the corresponding equations (4).

It has to be assumed, however, that the principal submatrix of $Q$ corresponding to the continuous variables is psd, since otherwise no convexification is possible. For the sake of brevity, we do not go into further details here.

# 3 On 0-1 QCQP with Quadratic Equations

In this section and the following three, we show how the QCR method can be extended to handle quadratic constraints. In this section, we deal with a relatively easy special case: that in which all variables are binary, and all quadratic constraints are equations. Throughout, we assume that the 0-1 QCQP instance has been written in the following form:

$$
\begin{aligned}
\min \quad & x^T Q^0 x + c^0 \cdot x && (7)\\
\text{s.t.} \quad & Ax = b && (8)\\
& Dx \le f && (9)\\
& x^T Q^j x + c^j \cdot x = h_j \quad (j = 1, \dots, m) && (10)\\
& x_i^2 - x_i = 0 \quad (i = 1, \dots, n). && (11)
\end{aligned}
$$

Any additional valid quadratic equations, such as (5) or (6), are assumed to have already been included in the system (10).

### 3.1 Semidefinite and Lagrangian relaxations

If we mechanically follow the scheme outlined in Section 2, we obtain the following semidefinite relaxation:

$$
\begin{aligned}
\inf \quad & Q^0 \bullet X + c^0 \cdot x\\
\text{s.t.} \quad & (8),\ (9)\\
& Q^j \bullet X + c^j \cdot x = h_j \quad (j = 1, \dots, m)\\
& \operatorname{diag}(X) = x\\
& Y \succeq 0.
\end{aligned}
$$

The corresponding Lagrangian relaxation is formed by relaxing all of the constraints (8)–(11), each with their own set of multipliers. We will denote these multipliers by $\lambda$, $\mu$, $\nu$ and $\phi$, respectively. The Lagrangian then takes the form:

$$f(x, \lambda, \mu, \nu, \phi) \;=\; x^T \bar{Q} x + \bar{c}^T x + \bar{h},$$

where:

$$
\begin{aligned}
\bar{Q} \;&=\; Q^0 + \sum_{j=1}^{m} \nu_j Q^j + \operatorname{Diag}(\phi)\\
\bar{c} \;&=\; c^0 + A^T \lambda + D^T \mu + \sum_{j=1}^{m} \nu_j c^j - \phi\\
\bar{h} \;&=\; -\lambda^T b - \mu^T f - \sum_{j=1}^{m} \nu_j h_j.
\end{aligned}
$$

The components of $\bar{Q}$ and $\bar{c}$ can be thought of as 'reduced costs' with respect to the given multipliers.

The Lagrangian dual is simply:

$$\sup_{\lambda,\mu,\nu,\phi} \; \inf_{x \in \mathbb{R}^n} f(x, \lambda, \mu, \nu, \phi).$$

### 3.2 Some remarks on the relaxations

Before presenting our reformulation scheme, we present some results concerning the relaxations presented in the previous subsection. We will find it helpful to write the dual of the SDP explicitly, as:

$$\begin{array}{ll} \sup & t \\ \text{s.t.} & \begin{pmatrix} \bar{h} - t & \bar{c}^T/2 \\ \bar{c}/2 & \bar{Q} \end{pmatrix} \succeq 0 \\ & \lambda, \nu, \phi, t \text{ free} \\ & \mu \geq 0. \end{array}$$

Here, $t$ is the dual variable for the constraint $Y_{00} = 1$ (which is implicit in the definition of $Y$), and $\bar{Q}$, $\bar{c}$ and $\bar{h}$ are as in the previous subsection.

From the result mentioned in Subsection 2.1, solving the Lagrangian dual is equivalent to solving the above dual SDP. The following two propositions show that the primal and dual SDPs behave quite differently:

**Proposition 1** *The primal SDP has a bounded feasible region, but it does not necessarily satisfy the Slater condition.*

**Proof.** For all $1 \leq i \leq n$, the matrix $Y$ contains the $2 \times 2$ principle submatrix $\begin{pmatrix} 1 & x_i \\ x_i & X_{ii} \end{pmatrix}$. Since $Y$ is psd, we have $X_{ii} \geq x_i^2$. Together with $x_i = X_{ii}$, this implies $x_i \in [0, 1]$ and $X_{ii} \in [0, 1]$ for all $i$. Moreover, since $Y$ is psd, it satisfies $b^T Y b \geq 0$ for all $b \in \mathbb{R}^{n+1}$. Setting $b = (-1/2, 1, 1, 0, \ldots, 0)^T$ we get

$$X_{11} + X_{22} + 2X_{12} - x_1 - x_2 + 1/4 \geq 0.$$

Together with $x_1 = X_{11}$ and $x_2 = X_{22}$, this implies $X_{12} \geq -1/8$. On the other hand, setting $b = (0, 1, -1, 0, \ldots, 0)^T$, we get $X_{11} + X_{22} - 2X_{12} \geq 0$. Since, as we have already seen, $X_{11} \leq 1$ and $X_{22} \leq 1$, we have $X_{12} \leq 1$. By symmetry, we then have $-1/8 \leq X_{ij} \leq 1$ for all pairs $(i, j)$. Therefore the feasible region is bounded.

To see that the primal SDP need not satisfy the Slater condition, just consider a 0-1 QCQP instance that contains the quadratic equation $x_k^2 = 0$ for some $k$. The corresponding constraint in the SDP, of the form $X_{kk} = 0$, prevents the matrix $Y$ from being positive definite. Therefore the primal SDP is not strictly feasible. $\square$

**Proposition 2** *The dual SDP satisfies the Slater condition, but its feasible region is unbounded.*

**Proof.** Given any values for $\lambda$, $\mu$ and $\nu$, we can obtain a feasible solution to the dual SDP by setting both $t$ and the components of $\phi$ to arbitrary large negative values. This shows that the feasible region is unbounded.

Moreover, if we set $t$ and the components of $\phi$ to large enough negative values, the matrix $\begin{pmatrix} \bar{h} - t & \bar{c}^T/2 \\ \bar{c}/2 & \bar{Q} \end{pmatrix}$ will be positive definite, making the dual solution strictly feasible. □

Proposition 1 implies that we can write 'min' instead of 'inf' in the primal SDP. More importantly, the two propositions imply the following result:

**Corollary 1** *The Lagrangian dual has the same value as the primal SDP, but there may not exist feasible Lagrangian multipliers that actually attain that value.*

The following instance, adapted from [5], illustrates this corollary:

$$\min \left\{ 2x_1x_2 - x_2^2 - 2x_1 : \, x_2^2 = 1, \, x \in \{0,1\}^2 \right\}.$$

One can easily show that the primal SDP gives a lower bound of $-1$, which is optimal, but that there is no dual solution (semidefinite or Lagrangian) that gives that lower bound.

Our next result indicates that including the equations (5) or (6) in the SDP could potentially cause problems:

**Proposition 3** *Suppose that a 0-1 QP or 0-1 QCQP instance contains linear equations of the form (8), and suppose that the SDP relaxation has been strengthened by including the equations (5) or (6). Then the primal SDP will not satisfy the Slater condition.*

**Proof.** We have already seen that the equation (6), together with $Y \succeq 0$, implies the equations (5). They in turn imply that

$$\begin{pmatrix} -b_j \\ a^j \end{pmatrix}^T Y \begin{pmatrix} -b_j \\ a^j \end{pmatrix} = 0$$

for all $j$. This implies that $Y$ is not positive definite, which means that the primal SDP is not strictly feasible. □

Fortunately, we have found that, despite the theoretical issues raised in this subsection, the supremum in the Lagrangian and semidefinite duals is almost always attainable in practice.

### 3.3 Reformulation

We now come to reformulation. By analogy with the original QCR method, we seek to transform a 0-1 QCQP instance into another 0-1 QCQP instance, that has the following five properties:

- The objective function is convex.

- The feasible region of the continuous relaxation is convex.
- The set of feasible solutions is identical to that of the original instance.
- The cost of each feasible solution is the same as it was in the original instance.
- The lower bound obtained by solving the continuous relaxation is equal to the lower bound obtained by applying semidefinite or Lagrangian relaxation to the original instance.

Note that the continuous relaxation of a 0-1 QCQP instance is obtained simply by replacing the constraints (11) with the constraint $x \in [0,1]^n$.

A first observation is that, when perturbing the objective function, we can exploit the equations (8), (10) and (11). Specifically, we are free to add terms of the form:

- $(\lambda^T A)x - \lambda^T b$ for some real vector $\lambda$ of appropriate dimension,
- $\nu_j(x^T Q^j x + c^j \cdot x - h_j)$ for some $\nu \in \mathbb{R}^m$
- $x^T \text{Diag}(\phi)x - \phi^T x$ for some $\phi \in \mathbb{R}^n$.

Note that, with this notation, the perturbed objective function will have the form of the Lagrangian $f(x, \lambda, \mu, \nu, \phi)$ defined in Subsection 3.1, except that $\mu$ will be zero. We will see that, in fact, we can fix $\lambda$ to zero as well, without affecting the quality of the reformulation.

A second observation is that, since the quadratic equations (10) are inherently non-convex, we will need to somehow convexify them, in such a way that the set of feasible solutions is unchanged. One simple way to do this is to replace each such equation with two quadratic inequalities, and then convexify each of the two inequalities independently, using the 'minimum eigenvalue' method of Hammer & Rubin [8], described in Subsection 2.3.

The following theorem shows that, using these two ideas, the desired reformulation can be obtained:

**Theorem 1** *Let a 0-1 QCQP instance of the form (7)-(11) be given. Suppose that the primal SDP described in Subsection 3.1 satisfies the Slater condition (and therefore that the supremum in the dual SDP is attainable). Then, suppose we perform the following four operations:*

- *solve the SDP and compute the optimal dual solution* $(\lambda^*, \mu^*, \nu^*, \phi^*)$*;*
- *perturb the objective function of the 0-1 QCQP instance by adding terms of the form* $x^T \text{Diag}(\phi^*)x - (\phi^*)^T x$ *and* $\nu_j^*(x^T Q^j x + c^j \cdot x - h_j)$ *for* $j = 1, \ldots, m$*;*
- *replace each quadratic equation with two quadratic inequalities;*

- *convexify the quadratic inequalities using the minimum eigenvalue method.*

*Then the reformulated instance will be convex, and the lower bound from its continuous relaxation will be equal to the lower bound from the SDP.*

**Proof.** The objective function of the reformulated instance is

$$x^T Q^* x + c^* \cdot x + h^*,$$

where:

$$\begin{aligned} Q^* &= Q^0 + \sum_{j=1}^{m} \nu_j^* Q^j + \mathrm{Diag}(\phi^*) \\ c^* &= c^0 + \sum_{j=1}^{m} \nu_j^* c^j - \phi^* \\ h^* &= -\sum_{j=1}^{m} \nu_j^* h_j. \end{aligned}$$

Note that $Q^*$ has the same form as the matrix $\bar{Q}$ described in Subsection 3.1. Since $\nu^*$ and $\phi^*$ belong to a feasible dual solution, we have $Q^* \succeq 0$. Therefore the objective function of the reformulated instance is convex. Moreover, the constraints in the reformulated instance are convex by assumption.

Now, from the equivalence of semidefinite and Lagrangian relaxation, the lower bound from the semidefinite relaxation is equal to:

$$\min \left\{ x^T \bar{Q}^* x + \bar{c}^* \cdot x + \bar{h}^* : \ x \in \mathbb{R}^n \right\}. \tag{12}$$

From the fact that this is a convex optimisation problem, and the fact that $\lambda^*$ and $\mu^*$ form part of an optimal dual solution, this bound is equal to:

$$\min \left\{ x^T Q^* x + c^* \cdot x + h^* : \ (8), \ (9), \ x \in \mathbb{R}^n \right\}.$$

The continuous relaxation of the reformulated instance is identical to this, except that it has some additional constraints; namely, the convexified quadratic inequalities and the constraint $x \in [0,1]^n$. The lower bound from the continuous relaxation is therefore at least as large as (12). Moreover, it cannot be larger, since the additional constraints are convex and are implied by constraints that have already been incorporated into the objective function of (12) with optimal multipliers. □

## 4 Handling Quadratic Inequalities

Now we consider how to extend the QCR method to general 0-1 QCQP, in which quadratic inequalities may be present. We assume that the 0-1

QCQP instance is of the form (7)-(11), but also has $r$ additional quadratic inequalities of the form:

$$x^T Q^j x + c^j \cdot x \leq h_j \qquad (j = m+1, \ldots, m+r). \tag{13}$$

We assume that the system (13) includes any valid quadratic inequalities that have been derived from linear inequalities using the methods mentioned in Subsection 2.2.

We will see that quadratic inequalities cause no problems when it comes to *relaxations*, but cause big difficulties when it comes to *reformulation*.

### 4.1 Semidefinite and Lagrangian relaxations

The semidefinite and Lagrangian relaxations are straightforward. To form the semidefinite relaxation, we simply add the following inequalities to the SDP described in Subsection 3.1:

$$Q^j \bullet X + c^j \cdot x \leq h_j \qquad (j = m+1, \ldots, m+r). \tag{14}$$

To form the Lagrangian relaxation, we simply relax the quadratic inequalities (13), using a vector of Lagrangian multipliers that we will call $\pi$. (Note that $\pi \in \mathbb{R}^r_+$.)

One can easily show that all of the results given in Subsection 3.2, i.e., Propositions 1 to 3, remain valid. Since the proofs carry through virtually unchanged, we omit the details.

### 4.2 Reformulation: Two difficulties

When it comes to reformulation, however, quadratic inequalities cause two serious difficulties.

The first is that we cannot always reformulate the 0-1 QCQP instance in such a way that the lower bound from the continuous relaxation is equal to the SDP bound. In fact, the best possible lower bound can be *much worse* than the SDP bound. This is illustrated by the following example:

**Example 1:** Consider the following 0-1 QCQP instance:

$$\begin{aligned} \min \quad & -\textstyle\sum_{i=1}^n x_i \\ \text{s.t.} \quad & x_i x_j \leq 0 \quad (1 \leq i < j \leq n) && (15) \\ & x_i^2 - x_i = 0 \quad (i = 1, \ldots, n). && (16) \end{aligned}$$

The optimal profit is $-1$, and the lower bounds from the primal and dual SDPs are easily shown to be $-1$ as well. Suppose, then, that we try to reformulate the instance. There is no point perturbing the objective function using the equations (16), since negative values for the multipliers $\phi_i$ would destroy convexity and positive values would weaken the lower bound. On the other hand, the quadratic inequalities (15) are non-convex, and therefore must be convexified using the equations (16). By symmetry, there exists an optimal reformulation in which the inequalities (15) are replaced by inequalities of the form:

$$\alpha(x_i^2 - x_i) + \alpha(x_j^2 - x_j) + x_i x_j \leq 0 \qquad (1 \leq i < j \leq n),$$

for some real $\alpha$. For convexity, we require $\alpha \geq 1/2$. The best lower bound is obtained when $\alpha = 1/2$. Then, the optimal solution $x^*$ to the continuous relaxation is $(1/2, \ldots, 1/2)^T$, yielding a lower bound of $-n/2$. □

The second difficulty is that finding an optimal reformulation is itself a rather complex optimisation problem. In principle, one could perturb each quadratic inequality in the same way as the objective function was perturbed in the previous section. That is, one could add terms of the following two kinds to the $k$th such inequality:

- $\nu_{jk}(x^T Q^j x + c^j \cdot x - h_j)$ for $j = 1, \ldots, m$, where $\nu_{jk} \in \mathbb{R}$;
- $x^T \text{Diag}(\phi^k) x - (\phi^k)^T x$, where $\phi^k \in \mathbb{R}^n$.

To find the best reformulation of this type, one would have to determine simultaneously the optimal values of the scalars $\nu_{jk}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, r$, the vectors $\phi^k$ for $k = 1, \ldots, r$, and the original vectors $\nu$ and $\phi$ associated with the objective function. This would have to be done in such a way that the perturbed objective and constraint functions were all convex.

It is possible to formulate the problem of finding the best such reformulation as one huge SDP. Unfortunately, the size and complexity of this SDP grows rapidly as $r$ increases. We do not go into further details, since we do not recommend such an approach.

### 4.3 Reformulation: Possible remedies

We now show that, if one is willing to convert a 0-1 QCQP instance into a *mixed* 0-1 QCQP instance (i.e., an instance in which continuous variables are present), then a strong reformulation can always be obtained:

**Proposition 4** *Let a 0-1 QCQP instance of the form (7)-(11), (13) be given. Suppose that the primal SDP described in Subsections 3.1 and 4.1 satisfies the Slater condition. Then, suppose we perform the following five operations:*

- *solve the SDP and compute the optimal dual solution* $(\lambda^*, \mu^*, \nu^*, \phi^*, \pi^*)$*;*

- *take the 0-1 QCQP instance and replace the quadratic inequalities (13) with the following constraints:*

$$x^T Q^j x + c^j \cdot x + s_j = h_j \quad (j = m+1, \ldots, m+r) \tag{17}$$
$$s_j \geq 0 \quad (j = m+1, \ldots, m+r), \tag{18}$$

  *where the $s_j$ are new continuous slack variables;*

- *perturb the objective function of the mixed 0-1 QCQP instance by adding terms of the form $\nu_j^*(x^T Q^j x + c^j \cdot x - h_j)$ for $j = 1, \ldots, m$, $x^T \text{Diag}(\phi^*)x - (\phi^*)^T x$, and $\pi_j^*(x^T Q^j x + c^j \cdot x + s_j - h_j)$ for $j = m+1, \ldots, m+r$;*

- *replace each quadratic equation with two quadratic inequalities;*

- *convexify the quadratic inequalities using the minimum eigenvalue method.*

*Then the reformulated mixed 0-1 QCQP instance will be convex, and the lower bound from its continuous relaxation will be equal to the SDP bound.*

**Proof.** The proof is similar to that of Theorem 1. The only significant difference is that the reformulated instance will contain an additional term of the form $\pi^* \cdot s$ in the objective function, along with the additional non-negativity constraint $s \geq 0$. Now, since $\pi \in \mathbb{R}_+^r$, the optimal solution to the continuous relaxation will always satisfy $s^* = 0$, so the presence of the $s$ variables has no effect on the quality of the lower bound. □

We illustrate Proposition 4 on the example given in Subsection 4.2:

**Example 1 (cont.):** Introducing slack variables, we obtain the following mixed 0-1 QCQP instance:

$$\begin{array}{lll} \min & -\sum_{i=1}^n x_i & \\ \text{s.t.} & x_i x_j + s_{ij} = 0 & (1 \leq i < j \leq n) \\ & x_i^2 - x_i = 0 & (i = 1, \ldots, n) \\ & s_{ij} \geq 0 & (1 \leq i < j \leq n). \end{array}$$

One can show that the optimal dual solution satisfies $\phi_i^* = 1$ for $i = 1, \ldots, n$ and $\pi_{ij}^* = 2$ for $1 \leq i < j \leq n$. We therefore perturb the objective function by adding

$$\sum_{i=1}^n (x_i^2 - x_i) + 2 \sum_{1 \leq i < j \leq n} (x_i x_j + s_{ij}).$$

We then replace each quadratic equation of the form $x_i x_j + s_{ij} = 0$ with two quadratic inequalities, of the form $x_i x_j + s_{ij} \leq 0$ and $-x_i x_j - s_{ij} \leq 0$.

Convexifying these inequalities using the minimum eigenvalue heuristic, we arrive (after some simplification) at the following reformulated instance:

$$
\begin{array}{lll}
\min & -2\sum_{i=1}^n x_i + \left(\sum_{i=1}^n x_i\right)^2 + 2\sum_{1\le i<j\le n} s_{ij} & \\
\text{s.t.} & \frac{1}{2}(x_i^2 - x_i) + \frac{1}{2}(x_j^2 - x_j) + x_i x_j + s_{ij} \le 0 & (1 \le i < j \le n) \\
& \frac{1}{2}(x_i^2 - x_i) + \frac{1}{2}(x_j^2 - x_j) - x_i x_j - s_{ij} \le 0 & (1 \le i < j \le n) \\
& x_i^2 - x_i = 0 & (i = 1, \dots, n) \\
& s_{ij} \ge 0 & (1 \le i < j \le n).
\end{array}
$$

Due to the form of the objective function of this instance, all optimal solutions to the continuous relaxation satisfy $\sum_{i=1}^n x_i = 1$ and $s = 0$. This yields a lower bound of $-1$, which is both optimal and equal to the SDP bound. □

The following proposition shows that, in fact, only *one* continuous slack variable needs to be added to the instance to obtain a strong reformulation:

**Proposition 5** *Let a 0-1 QCQP instance of the form (7)-(11), (13) be given, and suppose as before that the primal SDP satisfies the Slater condition. Then, suppose we perform the following five operations:*

- *solve the SDP and compute the optimal dual solution* $(\lambda^*, \mu^*, \nu^*, \phi^*, \pi^*)$*;*
- *add the following two constraints to the 0-1 QCQP instance:*

$$\sum_{j=m+1}^{m+r} \pi_j^* (x^T Q^j x + c^j \cdot x) + \tilde{s} = \sum_{j=m+1}^{m+r} \pi_j^* h_j \tag{19}$$

$$\tilde{s} \ge 0, \tag{20}$$

  *where* $\tilde{s}$ *is a new continuous slack variable;*

- *perturb the objective function of the mixed 0-1 QCQP instance by adding terms of the form* $\nu_j^*(x^T Q^j x + c^j \cdot x - h_j)$ *for* $j = 1, \dots, m$, $x^T \mathrm{Diag}(\phi^*) x - (\phi^*)^T x$, $\pi_j^*(x^T Q^j x + c^j \cdot x - h_j)$ *for* $j = m+1, \dots, m+r$, *and* $\tilde{s}$*;*
- *replace each quadratic equation with two quadratic inequalities;*
- *convexify all quadratic inequalities using the minimum eigenvalue method.*

*Then the reformulated mixed 0-1 QCQP instance will be convex, and the lower bound from its continuous relaxation will be equal to the SDP bound.*

**Proof.** The proof is similar to that of Theorem 1. The only significant difference is that the reformulated instance will contain the additional term $\tilde{s}$ in the objective function, along with the additional non-negativity constraint $\tilde{s} \ge 0$. Clearly, the optimal solution to the continuous relaxation will always satisfy $\tilde{s} = 0$, so the presence of $\tilde{s}$ has no effect on the quality of the lower bound. □

We illustrate Proposition 5 on the same example:

**Example 1 (cont.):** We add the constraint

$$2 \sum_{1 \le i < j \le n} x_i x_j + \tilde{s} = 0 \tag{21}$$

to the problem, along with the non-negativity constraint $\tilde{s} \ge 0$. We then perturb the objective function by adding

$$\sum_{i=1}^{n} (x_i^2 - x_i) + 2 \sum_{1 \le i < j \le n} x_i x_j + \tilde{s}.$$

We then replace the quadratic equation (21) with two quadratic inequalities. Convexifying these inequalities using the minimum eigenvalue heuristic, we arrive (after some simplification) at the following reformulated mixed 0-1 QCQP instance:

$$\begin{array}{llr}
\min & -2\sum_{i=1}^{n} x_i + \left(\sum_{i=1}^{n} x_i\right)^2 + \tilde{s} & \\
\text{s.t.} & \frac{1}{2}(x_i + x_j)^2 - \frac{1}{2}(x_i + x_j) \le 0 & (1 \le i < j \le n) \\
& \sum_{i=1}^{n}(x_i^2 - x_i) + 2\sum_{1 \le i < j \le n} x_i x_j + \tilde{s} \le 0 & \\
& (n-1)\sum_{i=1}^{n}(x_i^2 - x_i) - 2\sum_{1 \le i < j \le n} x_i x_j - \tilde{s} \le 0 & \\
& x_i^2 - x_i = 0 & (i = 1, \dots, n) \\
& \tilde{s} \ge 0. &
\end{array}$$

Due to the form of the objective function of this instance, all optimal solutions to the continuous relaxation satisfy $\sum_{i=1}^{n} x_i = 1$ and $\tilde{s} = 0$. This yields a lower bound of $-1$, as before. □

# 5 Integer Variables

Next, we consider how to deal with variables that are constrained to take integer values. We assume without loss of generality that such variables must be non-negative.

A first observation is that integer variables present no problem for the semidefinite and Lagrangian approaches. Indeed, an integer variable $x_i$ can be handled in the semidefinite relaxation simply by appending the inequality $x_i \ge 0$ to the relaxation, instead of the equation $x_i = X_{ii}$. Similarly, it can be handled in the Lagrangian relaxation simply by relaxing the linear inequality $x_i \ge 0$ (with a non-negative Lagrangian multiplier), rather than the quadratic equation $x_i^2 - x_i = 0$.

As for reformulation, note that we cannot use terms of the form $x_i^2 - x_i$ to perturb the objective or constraint coefficient of integer variables. This can prevent the existence of a convex reformulation. As a trivial example, consider the problem:

$$\min \left\{ x_1 x_2 - x_1^2 - x_2^2 : \; x_1 + x_2 = 2, \; x \in \mathbb{Z}_+^2 \right\}.$$

The fact that the objective function cannot be convexified is seen by considering the feasible solutions $(0, 2)$, $(1, 1)$ and $(2, 0)$, with costs $-4$, $-1$ and $-4$, respectively.

To get around this, we use the same approach as in [2], i.e., binary expansion. Let $x_i$ be an integer variable and suppose that we are given an associated upper bound, say $u_i \in \mathbb{Z}_+$, on the value taken by $x_i$ in any feasible solution. Let $r_i$ denote the number of binary bits needed to represent $u_i$, namely $\lceil \log_2 u_i \rceil$. Then, one can easily replace $x_i$ with a family of binary variables $\tilde{x}_i^1, \ldots, \tilde{x}_i^{r_i}$, using the substitution

$$x_i = \sum_{t=1}^{r_i} 2^{t-1} \tilde{x}_i^t.$$

In this way, problems with integer variables can be converted into 0-1 QCQP instances, to which the methods already explained above can be applied.

The following proposition states that binary expansion never causes the primal SDP bound to get any worse:

**Proposition 6** *Consider an SDP of the form:*

$$\begin{array}{lll} \inf & Q^0 \bullet X + c^0 \cdot x & \\ s.t. & Q^j \bullet X + c^j \cdot x = h_j & (j = 1, \ldots, m) \\ & Q^j \bullet X + c^j \cdot x \leq h_j & (j = m+1, \ldots, m+r) \\ & 0 \leq x_i \leq u_i & (i = 1, \ldots, q) \\ & X_{ii} = x_i & (i = q+1, \ldots, n) \\ & Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, & \end{array}$$

*and suppose that it is a relaxation of an optimisation problem in which the first $q$ variables are integer and the remaining variables are binary. Suppose that we apply binary expansion to the SDP. That is, we replace:*

- $x_i$ *with* $\sum_{t=1}^{r_i} 2^{t-1} \tilde{x}_i^t$ *for* $i = 1, \ldots, q$,
- $X_{ik}$ *with* $\sum_{t=1}^{r_i} 2^{t-1} \tilde{X}_{ik}^t$ *for* $i = 1, \ldots, q$ *and* $k = q+1, \ldots, n$,
- $X_{ki}$ *with* $\sum_{t=1}^{r_i} 2^{t-1} \tilde{X}_{ki}^t$ *for* $i = 1, \ldots, q$ *and* $k = q+1, \ldots, n$,
- $X_{ii}$ *with* $\sum_{t=1}^{r_i} \sum_{s=1}^{r_i} 2^{t-1} 2^{s-1} \tilde{X}_{ii}^{t,s}$ *for* $i = 1, \ldots, q$;

*we add the constraints*

$$\tilde{X}_{ii}^{t,t} = \tilde{x}_i^t \qquad (i = 1, \ldots, q;\ t = 1, \ldots, r_i);$$

*and, instead of imposing psd-ness on the matrix $Y$, we impose it on the expanded matrix*

$$\tilde{Y} = \begin{pmatrix} 1 & \tilde{x}^T \\ \tilde{x} & \tilde{X} \end{pmatrix},$$

*where*

$$\tilde{x} = (\tilde{x}_1^1, \ldots, \tilde{x}_1^{r_1}, \ldots, \tilde{x}_q^1, \ldots, \tilde{x}_q^{r_q}, x_{q+1}, \ldots, x_n)^T,$$

*and $\tilde{X}$ is the corresponding matrix. Then, the lower bound from the second SDP is no smaller than the one from the original SDP.*

**Proof.** It suffices to show that, given any feasible solution of the second SDP, there exists a feasible solution of the original SDP of the same value. To this end, let $(\tilde{x}^*, \tilde{X}^*)$ be a feasible solution to the second SDP, let $\tilde{Y}^*$ be the corresponding matrix, let $(x^*, X^*) \in \mathbb{R}^{n+n^2}$ be the corresponding pair defined by the mapping above, and let $Y^*$ be the corresponding matrix. By construction, $(x^*, X^*)$ satisfies all of the constraints in the original SDP, and has the same cost as $(\tilde{x}^*, \tilde{X}^*)$. It therefore only remains to be shown that $Y^*$ is psd. This is equivalent to showing that $b^T Y^* b \geq 0$ for all real vectors $b$ of dimension $n+1$. So, let $b = (b_0, \ldots, b_n)^T$ be such a vector and construct an expanded vector $\tilde{b}$ as follows. For $i = 1, \ldots, q$, we replace the single component $b_i$ with the following $r_i$ components:

$$b_i, 2b_i, \ldots, 2^{r_i - 1} b_i.$$

Now, since $\tilde{Y}^*$ is psd by assumption, we have $\tilde{b}^T \tilde{Y}^* \tilde{b} \geq 0$. From the way in which $Y^*$ was constructed from $\tilde{Y}^*$, this implies that $b^T Y^* b \geq 0$. □

The following example shows that, perhaps surprisingly, binary expansion can lead to an *improvement* in the lower bound:

**Example 2:** Consider the following instance with just one integer variable:

$$\min \left\{ x_1^2 - 3x_1 :\ x_1 \leq 3,\ x_1 \in \mathbb{Z}_+ \right\}.$$

There are two optimal solutions of cost $-2$, obtained by setting $x_1$ to either 1 or 2. The basic SDP relaxation is:

$$\min \left\{ X_{11} - 3x_1 :\ 0 \leq x_1 \leq 3,\ Y \succeq 0 \right\}.$$

The optimal solution to this relaxation is to set $x_1$ to 1.5 and $X_{11}$ to 2.25, giving a lower bound of $-2.25$. Now, applying binary expansion to this instance, we use the substitution $x_1 = \tilde{x}_1^1 + 2\tilde{x}_1^2$, where $\tilde{x}_1^1$ and $\tilde{x}_1^2$ are new binary variables. The SDP relaxation for the transformed instance is:

$$\begin{array}{ll} \min & \tilde{X}_{11}^{1,1} + 4\tilde{X}_{11}^{2,2} + 4\tilde{X}_{11}^{1,2} - 3\tilde{x}_1^1 - 6\tilde{x}_1^2 \\ \text{s.t.} & \tilde{x}_1^1 = \tilde{X}_{11}^{1,1} \\ & \tilde{x}_1^2 = \tilde{X}_{11}^{2,2} \\ & \tilde{Y} \succeq 0. \end{array}$$

One can check with an SDP solver that all optimal solutions to this SDP have cost $-2$. So, the transformed SDP yields an improved lower bound. □

Together with the results of the previous two sections, we have the following corollary:

**Corollary 2** *Given any integer quadratically constrained quadratic program, along with explicit upper bounds for each variable, we can construct in polynomial time an equivalent convex mixed 0-1 QCQP instance, such that the lower bound obtained by solving its continuous relaxation is no worse than the lower bound obtained by solving the SDP relaxation of the original instance.*

Now suppose, however, that the upper bounds $u_i$ are not given in advance. In this case, we run into a technical difficulty:

**Proposition 7** *Given an arbitrary MIQCQP instance, it is impossible to compute finite upper bounds on the values taken by the integer variables in finite time.*

**Proof.** It was shown by Jeroslow [10] that testing feasibility of an integer program with quadratic constraints is *undecidable*. This means that no finite algorithm for the problem exists. Now, if upper bounds on the values of integer variables could be computed in finite time, a finite algorithm for the feasibility problem could be obtained by simply enumerating all integer vectors whose values lie within the computed bounds, and checking them for feasibility. This is a contradiction. □

Fortunately, in practical applications, such upper bounds are likely to be readily available, since integer-constrained decision variables typically represent production quantities or the like.

# 6 Continuous Variables

In Subsection 4.3, we considered the use of continuous slack variables as a device to obtain strong reformulations. In this section, we consider how to deal with continuous variables in general. That is, we are concerned with general *mixed integer quadratically constrained quadratic programming* (MIQCQP). As in the previous section, we assume without loss of generality that all continuous variables must be non-negative.

In the context of the semidefinite and Lagrangian approaches, continuous variables can be handled in exactly the same way as integer variables, and therefore they present no problems. When it comes to reformulation, on the other hand, continuous variables cause serious difficulties. Indeed, a convex reformulation may not exist in general, and we cannot get around this issue by 'binarisation', as we did for integer variables.

It turns out to be helpful to distinguish several cases, which are dealt with in the following three propositions.

**Proposition 8** *Suppose an MIQCQP instance has explicit upper bounds for all integer variables, and that, for all continuous variables, all quadratic objective and constraint coefficients are zero. Then we can compute in polynomial time an equivalent mixed 0-1 QCQP instance that is convex, and whose continuous relaxation yields a lower bound that is at least as good as the SDP bound.*

**Proof.** From the results of the previous section, we can replace all integer variables with binary variables, without causing any worsening of the SDP bound. The proof is then similar to that of Theorem 1 and Propositions 4 and 5. The only significant difference is that for a continuous variable $x_i$, there is no constraint $x_i^2 - x_i = 0$, and therefore there is no associated dual variable $\phi_i$. Instead, there is the constraint $x_i \geq 0$, which can be retained in the reformulated instance. □

**Proposition 9** *Suppose an MIQCQP instance has explicit upper bounds for all integer variables, let $m$ and $r$ denote the number of quadratic equations and quadratic inequalities, respectively, and let $Q^0, \ldots, Q^{m+r}$ denote the objective and constraint matrices, as in the previous sections. Let $\hat{Q}^j$, for $j = 0, \ldots, m+r$, denote the principal submatrix of $Q^j$ corresponding to the continuous variables. If $\hat{Q}^j$ contains a non-zero entry, for some $1 \leq j \leq m$, or $\hat{Q}^j$ is not psd, for some $j \in \{0\} \cup \{m+1, \ldots, m+r\}$, then a convex reformulation may not exist.*

**Proof.** Suppose that $\hat{Q}^j$ contains a non-zero entry for some $1 \leq j \leq m$, but that $\hat{Q}^k$ is the zero matrix for all $k \neq j$. Then, regardless of how we perturb the $j$th quadratic equation, the submatrix $\hat{Q}^j$ will remain unchanged and the equation will remain non-convex. Moreover, even if we replace it with two quadratic inequalities, at least one of the inequalities will be non-convex, since it is impossible for both $\hat{Q}^j$ and $-\hat{Q}^j$ to be psd simultaneously when $\hat{Q}^j$ is non-zero.

Similarly, suppose that $\hat{Q}^j$ is not psd for some $m+1 \leq j \leq m+r$, but that $\hat{Q}^k$ is the zero matrix for all $k \neq j$. Then, regardless of how we perturb the corresponding quadratic inequality, the submatrix $\hat{Q}^j$ will remain unchanged and the inequality will remain non-convex. A similar argument applies when $\hat{Q}^0$ is not psd. □

**Proposition 10** *Suppose an MIQCQP instance has explicit upper bounds for all integer variables, and let $m$, $r$, $Q^0, \ldots, Q^{m+r}$ and $\hat{Q}^0, \ldots, \hat{Q}^{m+r}$ be as defined in the previous proposition. If $\hat{Q}^j$ is the zero matrix for $j = 1, \ldots, m$, and $\hat{Q}^j$ is psd for $j \in \{0\} \cup \{m+1, \ldots, m+r\}$, then we can compute in polynomial time an equivalent convex mixed 0-1 QCQP instance, but there may not exist a reformulated instance whose continuous relaxation yields a lower bound that is as strong as the SDP bound.*

**Proof.** In this case, we can convexify the instance by replacing each quadratic equation with two quadratic inequalities, and then convexifying all objective and constraint functions by adding large multiples of the terms $x_i^2 - x_i$ for all 0-1 variables.

On the other hand, we cannot convert quadratic inequalities into equations by introducing slack variables, as done in Subsection 4.3 for the pure 0-1 case, since this would take us into the situation described in the previous proposition. This means that quadratic inequalities cannot be used to perturb the objective function, and therefore we may not be able to obtain a strong reformulation. □

As an illustration of Proposition 9, it suffices to take the trivial instance

$$\min \left\{ x_1 + x_2 : \; x_1^2 + x_2^2 = 1, \; x \in \mathbb{R}^2_+ \right\}.$$

The following, more complex, example illustrates Proposition 10:

**Example:** Consider the following mixed 0-1 QCQP instance:

$$\begin{array}{lll} \min & -\sum_{i=1}^n x_i & \\ \text{s.t.} & x_i x_j + x_{n+1}^2 \leq 0 & (1 \leq i < j \leq n) \\ & x_i^2 - x_i = 0 & (i = 1, \ldots, n), \\ & x_{n+1} \geq 0. & \end{array}$$

Note that it is identical to the instance given in Subsection 4.2, apart from the extra terms involving the continuous variable $x_{n+1}$. The optimal profit is again $-1$, and the lower bounds from the primal and dual SDPs are easily shown to be $-1$ as well.

We cannot add slack variables to the quadratic inequalities, since this would yield quadratic equations involving the term $x_{n+1}^2$, which could not be convexified. On the other hand, if we keep the original variables, exactly the same argument as before shows that the optimal reformulation is obtained by replacing the quadratic inequalities with inequalities of the form:

$$\frac{1}{2}(x_i^2 - x_i) + \frac{1}{2}(x_j^2 - x_j) + x_i x_j + x_{n+1}^2 \leq 0.$$

The optimal solution $x^*$ to the continuous relaxation is $(1/2, \ldots, 1/2, 0)^T$, yielding a lower bound of $-n/2$ as before. □

**Remark:** It should not be surprising that continuous variables can prevent the existence of a convex reformulation. Indeed, non-convex QCQP, a purely continuous problem, is an $\mathcal{NP}$-hard global optimisation problem. If non-convex QCQP instances could be converted into convex QCQP instances in polynomial time, then $\mathcal{P}$ would equal $\mathcal{NP}$.

## 7 Conclusion

Although Lagrangian and semidefinite relaxations of QCQP are well known, their use as a tool for *reformulation* is less widely known. In this paper, we have shown that the Quadratic Convex Reformulation (QCR) technique of Billionnet *et al.* [2, 3] can be extended, under certain conditions, from the case of MIQP to the much more general case of MIQCQP. In most cases, we are able to obtain convex reformulations whose associated lower bounds are at least as good as the SDP bound, and sometimes better. A serious problem is however posed by continuous variables that have quadratic terms in one or more constraints. Such variables can cause convex reformulations to be weak, or even prevent them from existing at all. This is not surprising, however, given that QCQP is already $\mathcal{NP}$-hard in the strong sense.

An interesting potential topic for future research is the extension of the QCR method to optimisation problems involving higher-order polynomials. Another is whether one could obtain better reformulations by using second-order conic constraints [1, 4], rather than convex quadratic constraints (which are less general).

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