On Integer Polytopes with Few Nonzero Vertices

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Abstract

We provide a simple description in terms of linear inequalities of the (dominant of the) convex hull of the nonnegative integer vectors that satisfy a given linear constraint and have sum of the components that does not exceed 2. This description allows the replacement of “weak” knapsack-type constraints by stronger ones in several ILP formulations for practical problems, including, e.g., train-unit assignment.

1 Introduction

Let \( n, b, k \in \mathbb{N} \) and \( a \in \mathbb{N}^n \) such that \( a_1 \geq a_2 \geq \cdots \geq a_n \). We consider the following polytope

\[
P := \text{conv} \left\{ x \in \mathbb{Z}_+^n : \sum_{j=1}^n a_j x_j \geq b, \sum_{j=1}^n x_j \leq k \right\}.
\]

restricting attention to the case in which \( P \) is non-empty, i.e., \( k a_1 \geq b \). The two linear inequalities defining \( P \), that define the feasible region of an integer knapsack in minimization form \([4]\) with an additional cardinality constraint, arise in several practical applications. For instance, in the problem considered in \([1]\), a train trip, whose timetable has already been fixed and for which at least \( b \) passenger seats are required, has to be covered by train units of \( n \) distinct types, the \( j \)th type having \( a_j \) seats available. Composition constraints impose that at most \( k \) train units (some of which may be of the same type) can be joined together to cover the trip.

The main purpose of studying \( P \) is to find alternative (stronger) inequalities to replace the (generally weak) knapsack inequality \( \sum_{j=1}^n a_j x_j \geq b \). For this reason, one may also be interested in the dominant of \( P \):

\[
\overline{P} := \{ x \in \mathbb{R}^n : \exists \overline{x} \in P \text{ such that } x \geq \overline{x} \},
\]

since all the inequalities in “\( \geq \)” form with nonnegative coefficients that are valid for \( P \) are also valid for \( \overline{P} \) and vice versa.

Given that optimization of a linear objective function over \( P \) is a (weakly) NP-hard problem, a complete description of \( P \) by linear constraints is in general fairly complex. On the other hand, such a description may be simple when the upper bound \( k \) on the sum of the components of \( x \) is a very small number. For instance, in case \( k = 1 \), the vertices of \( P \) are precisely the unit vectors \( x \) with \( x_i = 1 \) for some \( i \) such that \( a_i \geq b \) and \( x_j = 0 \) for \( j \neq i \). Recalling that the \( a_j \) values are sorted in decreasing order, and letting \( g \in \{1, \ldots, n\} \) be such that \( a_g \geq b, a_{g+1} < b \) (with \( g := n \) if \( a_n \geq b \)), we have the following trivial descriptions for the case \( k = 1 \):

\[
P = \left\{ x \in \mathbb{R}_+^n : \sum_{j=1}^g x_j = 1 \right\}, \quad \overline{P} = \left\{ x \in \mathbb{R}_+^n : \sum_{j=1}^g x_j \geq 1 \right\}.
\]
For the case \( k = 3 \) a complete description of \( P \) or \( \overline{P} \) appears to be fairly complex, as it involves inequalities with arbitrarily large coefficients. In this note, we provide a simple complete description of \( \overline{P} \) for the case \( k = 2 \), showing that \( n \) inequalities with coefficients in \( \{0, 1, 2\} \) suffice.

2 The Case \( k = 2 \)

Let \( g \) be such that \( s^g \geq b \) and \( s^{g+1} < b \) (with \( g := 0 \) if \( a_1 < b \) and \( g := n \) if \( s^n \geq b \), \( t \) be such that \( 2a_t \geq b \) and \( 2a_{t+1} < b \) (with \( t := n \) if \( 2a_n \geq b \), and, for each \( i = g + 1, \ldots, t \), \( f(i) \) be such that \( a_i + a_{f(i)} \geq b \) and \( a_i + a_{f(i)+1} < b \) (with \( f(i) := n \) if \( a_i + a_n \geq b \) and \( f(t + 1) := t \)).

**Theorem 1** For the case \( k = 2 \),

\[
\overline{P} = \left\{ x \in \mathbb{R}^n_+ : \sum_{j=1}^{i-1} 2x_j + \sum_{j=i}^{f(i)} x_j \geq 2, \quad i = g + 1, \ldots, t + 1 \right\}.
\] (1)

**Proof** We first show that the inequalities in (1), together with the non-negativity constraints and the integrality condition, yield a valid description of the integer points in \( \overline{P} \), i.e.,

\[
\overline{P} = \text{conv} \left\{ x \in \mathbb{Z}^n_+ : \sum_{j=1}^{i-1} 2x_j + \sum_{j=i}^{f(i)} x_j \geq 2, \quad i = g + 1, \ldots, t + 1 \right\}.
\] (2)

Indeed, the integer points in \( P \) coincide with the vectors \( x \in \mathbb{Z}^n_+ \) with either one component \( x_i > 0 \) for some \( i \leq g \) and sum of the components not larger than 2, or with one component \( x_i > 0 \) for some \( g < i \leq t \), and either \( x_i = 2 \) or \( x_i = 1 \) and \( x_i = 1 \) for some \( j \leq f(i) \). The integer points in \( \overline{P} \) are those that componentwise dominate one of these vectors. In order to verify (2), it is sufficient to check that all the integer points in \( \overline{P} \) satisfy the inequalities listed, and that all the integer points not in \( \overline{P} \) violate at least one of these inequalities. This is easily done by a tedious case analysis that is omitted here.

Having proved (2), what remains to be shown is that all the vertices of the polyhedron, say \( Q \), on the right-hand side of (1) are integer. Write the inequalities on the right-hand side of (1) in the compact form \( Cx \geq d \) (not including the non-negativity constraints). Given a vertex \( v \) of \( Q \), let \( u \) be the subvector of \( v \) containing the strictly positive components of \( v \) and \( m \) be the number of components of \( u \). It is well known (see, e.g., [5]) that there exists an \( m \times m \) non-singular submatrix \( S \) of \( C \) and a corresponding subvector \( r \) of \( d \) with \( m \) components such that \( Su = r \). In our application, all components of \( r \) are equal to 2. Furthermore, the entries of \( S \) are in \( \{0, 1, 2\} \) according to the following pattern:

![Pattern Diagram](image)

Now apply the following operations to the equation system \( Su = r \) (in the given order):

1. Subtract the first column from the second column.
2. Subtract the second column from the third column.
3. Subtract the third column from the fourth column.
1. subtract the last equation from the first \( m - 1 \) equations:

2. multiply the variables corresponding to the blocks I and II by \(-1\):

3. multiply the last equation by \(-1\).

Then we obtain an equivalent system \( S'u' = r' \), where \( S' \) has the following form:

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<th>I</th>
<th>II</th>
<th>III</th>
</tr>
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<tbody>
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<td>1</td>
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<tr>
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</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td>2</td>
</tr>
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```

Observe that block II has at most one column, as otherwise there would be two identical columns in \( S' \), i.e., \( S' \) and \( S \) would be singular. If block II is empty, then we can divide the last equation by 2, yielding the equivalent system \( S''u'' = r'' \), with \( S'' \) totally unimodular since it is a 0-1 matrix where the ones occur consecutively in the rows. This implies that \( u' \) and \( u \) are integer. If block II is a single column, expanding \( \det S' \) along this single column gives \( \det S' = \pm \det S''' \), where \( S''' \) is the matrix obtained from \( S' \) by removing the last row and the column in block II. As \( S''' \) is totally unimodular, being a 0-1 matrix where the ones occur consecutively in the rows, again we have that \( u' \) and \( u \) are integer.

\[ \square \]

**Example**

In order to illustrate the above result, let us consider the numerical example, taken from a case study in [1], in which \( n = 8, b = 1302 \) and \( a = (1150, 1044, 786, 702, 543, 516, 495, 360) \). In this case we have \( g = 0, t = 4, f(1) = f(2) = 8, f(3) = 6, f(4) = 4 \), leading to the following constraints:

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 & \geq 2 \\
2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 & \geq 2 \\
2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 & > 2 \\
2x_1 + 2x_2 + 2x_3 + x_4 & > 2 \\
2x_1 + 2x_2 + 2x_3 + 2x_4 & > 2
\end{align*}
\]

out of which the second is dominated by the first and the last is dominated by the last but one.

### 3 Extensions and Open Problems

Sticking to the case \( k = 2 \), there are a few generalizations or variants in which the description of the convex hull may be simple as well, i.e., with linear inequalities whose coefficients are in \( \{0, 1, 2\} \). In particular, the following open problems may be investigated in the future:

- Provide a complete description of \( P \) (as opposed to \( \overline{P} \)).
• Provide a complete description of the counterpart of $P$ (or $\overline{P}$) for the case in which the additional upper bounds $x_j \leq 1$ ($j = 1, \ldots, n$) are imposed.

• Provide a complete description of the counterpart of $P$ (or $\overline{P}$) for the case in which the knapsack inequality is in “$\leq$” form, i.e., $\sum_{j=1}^{n} a_j x_j \leq b$.

In fact, a problem that generalizes all those above and still may have a simple solution is the following. Consider a set of $m$ vectors $x^1, \ldots, x^m \in \{0,1,2\}^n$ such that $\sum_{j=1}^{n} x^i_j \leq 2$ for $i = 1, \ldots, m$. Provide a complete description of (the dominant of) $\text{conv}\{x^1, \ldots, x^m\}$. Note that, in case each vector is binary and has sum of the components is exactly 2, the set of vectors corresponds to an undirected graph with vertex set $\{1, \ldots, n\}$ and $m$ edges, the $i$th, associated with the vertex $x^i$, joining the two vertices $j$ and $k$ for which $x^i_j = x^i_k = 1$.

Acknowledgments
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References


